

CUSUM TEST : A PREDICTIVE TEST OF STRUCTURAL CHANGE

RECURSIVE RESIDUALS

CONSIDER THE MODEL

$$y_t = \sum_{j=1}^k x_{jt} \beta_j + u_t$$

$$y_t = x_t \beta + u_t$$

$t = 1, 2, \dots, n$

$$x_t = \begin{pmatrix} x_{1t} \\ x_{2t} \\ \vdots \\ x_{kt} \end{pmatrix}' = (x_{1t} \ x_{2t} \ \dots \ x_{kt})$$
$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix}$$

AT ANY GIVEN t , DEFINE

$$X_{t-1} = \begin{bmatrix} x_{11} & \dots & x_{k1} \\ x_{12} & & x_{k2} \\ \vdots & & \vdots \\ x_{1,t-1} & & x_{k,t-1} \end{bmatrix}$$

$$y^{(t-1)} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{t-1} \end{bmatrix}$$

GET THE OLS ESTIMATOR

$$\hat{\beta}_{(t-1)} = (X_{t-1}' X_{t-1})^{-1} X_{t-1}' y^{(t-1)}$$

NOW USE THIS $\hat{\beta}_{(t-1)}$ ALONG WITH
THE ACTUAL $x_t = (x_{1t} \ x_{2t} \ \dots \ x_{kt})$

TO GET A FORECAST FOR y_t

$$y_{ft} = x_t \hat{\beta}_{(t-1)}$$

THE FORECAST YIELDS THE PREDICTION ERROR

$$e_t = y_t - x_t(\beta_{t-1})$$

NOTE THAT $E(e_t) = 0$ (AS SHOWN EARLIER)

$$\text{Var}(e_t) = \sigma^2 \left(1 + x_t' (x_{t-1}' x_{t-1})^{-1} x_t' \right)$$

DEFINE
$$w_t = \frac{e_t}{\sqrt{\text{Var}(e_t)}}$$

UNDER H_0 THAT THE COEFFICIENTS REMAIN STABLE OUT OF SAMPLE $w_t \sim N(0, \sigma^2)$

THUS
$$\frac{w_t}{\sigma} \sim N(0, 1)$$

CUSUM TEST :

CONSIDER THE CUMULATIVE SUM OF RESIDUALS

$$W_t = \sum_{r=k+1}^t \frac{w_r}{\sigma}$$

HER
$$\hat{\sigma}^2 = \frac{1}{n - (k+1)} \sum_{r=k+1}^n (w_r - \bar{w})^2$$

$$\bar{w} = \frac{1}{n - k} \sum_{r=k+1}^n w_r$$

NOTE
$$\frac{w_r}{\sigma} \sim N(0, 1)$$

THUS W_t HAS AN APPROXIMATE NORMAL DISTRIBUTION WITH MEAN 0 & VARIANCE $(t - (k+1))$.

BROWN, DURBIN, & EVANS DEVELOPED CONFIDENCE BOUNDS OF THE CUMULATIVE SUM

BY PLOTTING 2 LINES THAT CONNECT

THE POINTS $(k, \pm a(n-k)^{1/2})$

& $(n \pm 3a(n-k)^{1/2})$.

FOR THE 95% LEVEL OF SIGNIFICANCE,

$\alpha = 0.948$.

THE CUMULATIVE SUM ~~IS~~ W_t IS

PLOTTED AGAINST t FOR $t \geq k$.

IF THE LINE STRAYS OUTSIDE THE BOUNDS, MODEL INSTABILITY IS IMPLIED.