

MORE MATRIX ALGEBRA

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CONSIDER THE EQUATIONS

$$a_{11}x_1 + a_{12}x_2 = \lambda x_1$$

$$a_{21}x_1 + a_{22}x_2 = \lambda x_2$$

①

IN MATRIX FORM,

$$Ax = \lambda x$$

$$\text{WHERE } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

CLEARLY IF x^0 SATISFIES
THIS EQUATION, SO DOES THE
VECTOR tx^0 FOR ANY t .

SO WE NORMALIZE BY THE RESTRICTION
 $x^T x = 1$.

NOTE THAT ① \Rightarrow

$$(a_{11} - \lambda)x_1 + a_{12}x_2 = 0$$

$$a_{21}x_1 + (a_{22} - \lambda)x_2 = 0$$

IN MATRIX FORM

$$Ax - \lambda x = (A - \lambda I)x = 0$$

FOR A NON-TRIVIAL SOLUTION (I.E., OTHER
THAN $x=0$) WE NEED THAT THE MATRIX
 $A - \lambda I$ MUST BE SINGULAR.

$$\text{THUS } \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

THIS IS A QUADRATIC EQUATION IN λ
WITH ROOTS λ_1 & λ_2 .

THIS IS KNOWN AS THE CHARACTERISTIC EQUATION
OF THE MATRIX A & THE ROOTS ARE ITS CHARACTERISTIC
ROOTS OR EIGEN VALUES.

EXAMPLE :

$$A = \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(4-\lambda) - 2 = 0$$

$$\lambda^2 - 9\lambda + 20 - 2 = 0 \Rightarrow \lambda^2 - 9\lambda + 18 = 0$$

$\lambda_1 = 6, \lambda_2 = 3$ ARE THE ROOTS.

CHARACTERISTIC VECTORS

FOR $\lambda_1 = 6$, $Ax = \lambda_1 x$

$$\Rightarrow \begin{cases} 5x_1 + x_2 = 6x_1 \\ 2x_1 + 4x_2 = 6x_2 \end{cases} \Rightarrow x_1 = x_2$$

FURTHER, $x^T x = 1 \Rightarrow x_1^2 + x_2^2 = 1 \Rightarrow 2x_1^2 = 1 \Rightarrow x_1 = x_2 = \frac{1}{\sqrt{2}}$

THE CHARACTERISTIC VECTOR ASSOCIATED WITH $\lambda_1 = 6$ IS

$$x^1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

FOR $\lambda_2 = 3$,

$$\begin{cases} 5x_1 + x_2 = 3x_1 \\ 2x_1 + 4x_2 = 3x_2 \end{cases} \Rightarrow \begin{cases} x_2 = -2x_1 \\ x_2^2 = 4x_1^2 \end{cases}$$

$$x^T x = 1 \Rightarrow \begin{cases} x_1^2 + x_2^2 = 1 \\ 5x_1^2 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 1/\sqrt{5} \\ x_2 = -2/\sqrt{5} \end{cases}$$

FOR $\lambda_2 = 3$, THE CORRESPONDING CHARACTERISTIC ROOT IS

$$x^2 = \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}$$

ASSUME THAT A IS A SYMMETRIC MATRIX OF ORDER R THEN A HAS R CHARACTERISTIC ROOTS (MAY NOT BE ALL DIFFERENT)

THE CHARACTERISTIC VECTORS SATISFY

$$Ax^1 = \lambda_1 x^1, Ax^2 = \lambda_2 x^2, \dots, Ax^R = \lambda_R x^R$$

THEN

$$x^1 A x^1 = x^1 (\lambda_1 x^1) = \lambda_1 x^1 x^1 = \lambda_1$$

$$x^2 A x^2 = x^2 (\lambda_2 x^2) = \lambda_2 x^2 x^2 = \lambda_2$$

$$x^k A x^k = x^k (\lambda_k x^k) = \lambda_k x^k x^k = \lambda_k$$

NOW CONSIDER 2 DIFFERENT VECTORS x^i & x^j .

$$A x^i = \lambda_i x^i$$

$$x^j A x^i = \lambda_i x^j x^i$$

FURTHER,

$$A x^j = \lambda_j x^j$$

$$x^i A x^j = \lambda_j x^i x^j$$

BY SYMMETRY OF A,

$$\begin{aligned} [x^j A x^i] &= x^i A x^j \\ &= x^i A x^j \end{aligned}$$

$$\begin{aligned} \text{THUS } [\lambda_j x^j x^i] &= \lambda_j x^i x^j \\ &= x^i A x^j = \lambda_i x^i x^j \end{aligned}$$

THIS IS TRUE FOR ANY PAIR (λ_i, λ_j) NOT NECESSARILY EQUAL TO ONE ANOTHER.

HENCE

$$\lambda_i x^i x^j = \lambda_j x^i x^j \Rightarrow x^i x^j = 0 \text{ FOR ANY PAIR } i \neq j$$

THE CHARACTERISTIC VECTORS ARE PAIRWISE ORTHOGONAL.

DIAGONALIZATION OF MATRICES

CONSIDER THE MATRIX

$$X = \begin{bmatrix} x^1 & x^2 & \dots & x^k \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

MATRIX

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \dots & \\ & & & \lambda_k \end{pmatrix}$$

& THE DIAGONAL

THEN

$$A X = A (x^1 \ x^2 \ \dots \ x^k)$$

$$= (A x^1 \ A x^2 \ \dots \ A x^k) = (\lambda_1 x^1 \ \lambda_2 x^2 \ \dots \ \lambda_k x^k)$$

 \Rightarrow

$$A X = X \Lambda$$

$$X' A X = X' X \Lambda = \Lambda \quad (\text{RECALL THAT } X' X = I)$$

THUS

$$\boxed{X' A X = \Lambda}$$

EXAMPLE:

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}; \quad |A - \lambda I| = 0$$

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$$\begin{vmatrix} (5-\lambda) & 2 \\ 2 & (5-\lambda) \end{vmatrix} = 0 \Rightarrow (5-\lambda)^2 - 4 = 0$$

$$\lambda^2 - 10\lambda + 25 - 4 = 0$$

$$\lambda^2 - 10\lambda + 21 = 0 \Rightarrow \lambda_1 = 7, \quad \lambda_2 = 3$$

$$Ax^1 = \lambda_1 x^1 \Rightarrow \begin{cases} 5x_1 + 2x_2 = 7x_1 \\ 2x_1 + 5x_2 = 7x_2 \end{cases} \quad \left| \quad x^1 = \begin{pmatrix} 1/\sqrt{2} \\ +1/\sqrt{2} \end{pmatrix} \right.$$

$$Ax^2 = \lambda_2 x^2 \Rightarrow \begin{cases} 5x_1 + 2x_2 = 3x_1 \\ 2x_1 + 5x_2 = 3x_2 \end{cases} \quad \left| \quad x^2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \right.$$

$$X = \begin{pmatrix} x^1 & x^2 \end{pmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \quad \text{VERIFY, } X'X = I$$

$$X'AX = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 7/\sqrt{2} & 3/\sqrt{2} \\ 7/\sqrt{2} & -3/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix}$$

CHARACTERISTIC ROOTS OF DEFINITE MATRICES

THE MATRIX OF CHARACTERISTIC VECTORS OF A IS
 X
 SATISFYING X'X = I
 THUS X' = X⁻¹ & XX' = XX⁻¹ = I

RECALL,

$$\Rightarrow \begin{aligned} X'AX &= \Lambda \\ XX'AXX' &= X\Lambda X' \\ \Rightarrow A &= X\Lambda X' \end{aligned}$$

HENCE, DEFINE $X'y = z$. THEN, $z'\Lambda z = y'Ay$

$$= \sum_{j=1}^n \lambda_j z_j^2 \geq 0$$

FOR ANY y IF $\lambda_j \geq 0$ FOR ALL j.
 BUT $y'Ay \geq 0 \Rightarrow A$ IS POSITIVE SEMI-DEFINITE.

DETERMINANT OF A SYMMETRIC MATRIX A

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RECALL THAT $A = X \Lambda X'$

$$\text{THUS } |A| = |X \Lambda X'| = |X| \cdot |\Lambda| \cdot |X'| = |X'| \cdot |X| \cdot |\Lambda|$$

$$\text{NOW, } |X'X| = |I| = 1$$

$$\text{ALSO } |X'X| = |X'| \cdot |X|$$

$$\text{HENCE, } |A| = |X'X| \cdot |\Lambda| = |\Lambda| = \prod_j \lambda_j$$

IF ANY ONE OF THE CHARACTERISTIC ROOTS OF A EQUALS 0, A IS SINGULAR.

TRACE OF A SYMMETRIC MATRIX

$$A = X \Lambda X'$$

$$\text{tr}(A) = \text{tr}(X \Lambda X') = \text{tr}(X'X) = \text{tr}(\Lambda) = \sum_j \lambda_j$$

TRACE OF A EQUALS THE SUM OF ITS CHARACTERISTIC ROOTS.

CHARACTERISTIC ROOTS OF A SYMMETRIC IDEMPOTENT MATRIX

LET A BE A SYMMETRIC & IDEMPOTENT MATRIX
 THEN $A' = A$ & $A'A = A.A = A$

LET λ_j BE THE j TH ROOT & x^j THE ASSOCIATED CHARACTERISTIC VECTOR. THEN

$$\text{HENCE } Ax^j = \lambda_j x^j$$

$$A(Ax^j) = A(\lambda_j x^j) = \lambda_j (Ax^j) = \lambda_j (\lambda_j x^j) = \lambda_j^2 x^j$$

$$\lambda_j x^j = \lambda_j^2 x^j \Rightarrow (\lambda_j - \lambda_j^2) x^j = 0$$

$$\lambda_j - \lambda_j^2 = 0 \Rightarrow \lambda_j = 0, 1$$

THUS ALL CHARACTERISTIC ROOTS OF A SYMMETRIC IDEMPOTENT MATRIX MUST EQUAL EITHER 0 OR 1.

RANK OF A SYMMETRIC IDEMPOTENT MATRIX A

$$A = X \Lambda X'$$

$$\begin{aligned} \text{RANK}(A) &= \text{RANK}(X \Lambda X') \\ &= \min [\text{RANK}(X), \text{RANK}(\Lambda)] \end{aligned}$$

NOTE $X'X = I$

$$\text{RANK}(X'X) = \text{RANK}(I) = n$$

BUT $\text{RANK}(X'X) = \text{RANK}(X) \Rightarrow \text{RANK}(X) = n$

HENCE $\text{RANK}(A) = \min(n, \text{RANK}(\Lambda))$

NOW, $\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}$

HENCE, $\text{RANK}(\Lambda) = \# \text{ OF NON-ZERO } \lambda_j \text{'S}$

THIS IS TRUE FOR ALL SYMMETRIC MATRICES.
 BUT IF A IS ALSO IDEMPOTENT, EACH λ_j IS EITHER 0 OR 1. IN THIS CASE, THE NUMBER OF NON-ZERO CHARACTERISTIC ROOTS = $\sum \lambda_j = \text{tr}(\Lambda)$

BUT $\text{tr}(\Lambda) = \text{tr}(A)$

HENCE THE RANK OF A = $\text{tr}(A)$

DISTRIBUTION OF A QUADRATIC FORM

CONSIDER A RANDOM VECTOR $u \sim N(0, I)$

LET $q = u' A u$ WHERE A IS SYMMETRIC

THEN $A = X \Lambda X'$

& $q = (u' X) \Lambda (X' u)$

DEFINE $y = X' u$

THEN $E(y) = X' E(u) = 0$
 $V(y) = E[yy'] = E[X' u u' X] = X' E(uu') X = X' X = I$

THUS $y \sim N(0, I)$

HENCE, $q = y' Ay = \sum_j \lambda_j y_j^2$ IS A LINEAR COMBINATION OF SQUARES ON STANDARD NORMAL VARIABLES (y_j) .

FURTHER, IF A IS SYMMETRIC & IDEMPOTENT, EACH λ_j IS EITHER 0 OR 1. SUPPOSE THAT p IS THE NUMBER OF NON-ZERO CHARACTERISTIC ROOTS. THEN q IS THE SUM OF SQUARES OF p RANDOM VARIABLES y_j EACH $N(0,1)$. [NOTE THAT p IS THE TRACE OF A AS SHOWN BEFORE.]

THEREFORE $q \sim \chi_p^2$.

INDEPENDENCE OF QUADRATIC FORMS

CONSIDER A RANDOM VECTOR x WITH $E(x) = 0$ & $V(x) = E[xx'] = I$ & TWO QUADRATIC FORMS

$$q_1 = x'Ax \quad \& \quad q_2 = x'Bx$$

ASSUME THAT BOTH A & B ARE IDEMPOTENT MATRICES (ALSO SYMMETRIC).

$$\text{THEN } q_1 = x'A'Ax \quad \& \quad q_2 = x'B'Bx$$

$$\text{DEFINE } x_1 = Ax \quad \& \quad x_2 = Bx$$

$$\text{THEN } q_1 = x_1'x_1 \quad \& \quad q_2 = x_2'x_2$$

$$\text{BECAUSE } E(x) = 0, \quad E(x_1) = AE(x) = 0 \quad \&$$

$$E(x_2) = BE(x) = 0$$

$$E[x_1 x_2'] = E[Ax x' B'] = AE(xx')B' = AB' = AB$$

THUS x_1 & x_2 ARE INDEPENDENT (AND HENCE

q_1 & q_2 ARE INDEPENDENT) IN $AB=0$.

NOW CONSIDER

$$x'x = \sum x_i^2 = \sum (x_i - \bar{x})^2 + n\bar{x}^2$$

DEFINE

$$q_1 = \sum (x_i - \bar{x})^2 = x' M_0 x$$

WHERE $M_0 = I - \frac{1}{n} z z'$ [AS SHOWN BEFORE]

BECAUSE $x \sim N(0, I)$, $q_1 \sim \chi_p^2$

WHERE $p = \text{tr}(M_0)$

BUT $\text{tr}(M_0) = \text{tr}\left[I - \frac{1}{n} z z'\right] = \text{tr}[I] - \frac{1}{n} \text{tr}[z z']$
 $= n - \frac{1}{n} \cdot n = n - 1$

THUS $q_1 \sim \chi_{n-1}^2$

NEXT DEFINE $q_2 = n\bar{x}^2 = n \cdot \frac{(\sum x_i)^2}{n^2}$

BUT $\sum x_i = z'x$. THUS $q_2 = n \frac{[x' z z' x]}{n^2}$

DEFINE THE VECTOR $j = \frac{1}{\sqrt{n}} z$

THEN $j j' = \frac{z z'}{n}$. & $q_2 = x' [j j'] x$

REFINE $P_0 = I - M_0 = I - \left(I - \frac{1}{n} z z'\right) = \frac{1}{n} z z' = [j j']$

THEN

$$q_2 = x' P_0 x$$

$M_0 P_0 = M_0 (I - M_0) = M_0 - M_0 M_0 = M_0 - M_0 = 0$

THUS q_1 & q_2 ARE INDEPENDENT.

$\text{tr}(P_0) = \text{tr}(I - M_0) = \text{tr}(I) - \text{tr}(M_0) = n - (n-1) = 1$

HENCE $q_1 = \sum (x_i - \bar{x})^2 \sim \chi_{n-1}^2$

& $q_2 = n\bar{x}^2 \sim \chi_1^2$

WHERE q_1 & q_2 ARE INDEPENDENT.

INDEPENDENCE OF A LINEAR & A QUADRATIC FORM

CONSIDER A QUADRATIC FORM

$$Q = x'Ax$$

& A VECTOR $p = Lx$

HERE A IS SYMMETRIC IDEMPOTENT MATRIX
& L IS A $K \times N$ MATRIX.

WE WANT TO KNOW UNDER WHAT CONDITION
THE SINGLE RANDOM VARIABLE Q WILL BE
INDEPENDENT OF EACH ELEMENT p_j ($j=1,2,\dots,n$) OF
THE VECTOR p .

BECAUSE A IS A SYMMETRIC IDEMPOTENT
MATRIX,

$$x'Ax = x'A'Ax$$

DEFINE $r = Ax$ THEN

$$Q = x'Ax = r'r$$

NOTE THAT $E(p) = LE(x) = 0$

$$\& E(r) = AE(x) = 0$$

THE COVARIANCE MATRIX BETWEEN p & r IS

$$E(pr') = E[Lxx'A'] = LE(xx')A'$$

BUT $x \sim N(0, I)$ & $E(xx') = I$

$$\text{HENCE } E(pr') = LA' = LA$$

IF ELEMENTS OF THE VECTORS p & r ARE
INDEPENDENT, $E(pr') = 0 \Rightarrow LA = 0$

OF COURSE WHEN p & r ARE INDEPENDENT,
BECAUSE $Q = r'r$, p & Q ARE ALSO
INDEPENDENT.

THE OLS COEFFICIENT VECTOR :

$$\hat{\beta} = \beta + (X'X)^{-1}X'u$$

$$E(\hat{\beta}) = \beta$$

$$\& V(\hat{\beta}) = \sigma^2(X'X)^{-1}$$

IF $u \sim N(0, \sigma^2 I)$, $\hat{\beta} \sim N(\beta, \sigma^2(X'X)^{-1})$

CONSIDER THE QUADRATIC FORM

$$\frac{e'e}{\sigma^2} = \frac{u'Mu}{\sigma^2} = \left(\frac{u}{\sigma}\right)' M \left(\frac{u}{\sigma}\right)$$

BECAUSE $u \sim N(0, \sigma^2 I)$, $\left(\frac{u}{\sigma}\right) \sim N(0, I)$

HENCE, $\frac{e'e}{\sigma^2} \sim \chi^2_{n-k}$

NOW CONSIDER THE LINEAR FORM

$$\frac{\hat{\beta} - \beta}{\sigma} = \frac{(X'X)^{-1}X'u}{\sigma} = (X'X)^{-1}X' \left(\frac{u}{\sigma}\right)$$

DEFINE $(X'X)^{-1}X' \equiv A$

THEN $\frac{\hat{\beta} - \beta}{\sigma} = A \left(\frac{u}{\sigma}\right)$

NOW
$$AM = (X'X)^{-1}X' [I - X(X'X)^{-1}X']$$
$$= (X'X)^{-1}X' - (X'X)^{-1}X'X(X'X)^{-1}X'$$
$$= (X'X)^{-1}X' - (X'X)^{-1}X' = \underline{0}$$

THEREFORE $\frac{\hat{\beta} - \beta}{\sigma}$ & $\frac{e'e}{\sigma^2}$ ARE

INDEPENDENT

ANY INDIVIDUAL ELEMENT OF $\frac{\hat{\beta} - \beta}{\sigma}$, SAY $\frac{\hat{\beta}_k - \beta_k}{\sigma}$ HAS THE VARIANCE s^{kk} WHERE s^{kk} IS THE k TH DIAGONAL ELEMENT OF $(X'X)^{-1}$.

HENCE
$$\left(\frac{\hat{\beta}_k - \beta_k}{\sigma} \right) \frac{1}{\sqrt{s^{kk}}} \sim N(0, 1)$$

&
$$\frac{(\hat{\beta}_k - \beta_k)^2}{\sigma^2 (s^{kk})} \sim \chi^2_1$$

FURTHER $\frac{e'e}{\sigma^2} \sim \chi^2_{n-k}$ & IS INDEPENDENT OF $\frac{\hat{\beta}_k - \beta_k}{\sigma \sqrt{s^{kk}}}$. THUS THE RATIO

$$\frac{\frac{\hat{\beta}_k - \beta_k}{\sigma \sqrt{s^{kk}}}}{\sqrt{\frac{e'e}{\sigma^2 (n-k)}}} = \frac{\frac{\hat{\beta}_k - \beta_k}{\sqrt{s^{kk} \hat{\sigma}^2}}}{\sqrt{\frac{e'e}{\sigma^2 (n-k)}}} \sim t_{n-k}$$

WHERE
$$\hat{\sigma}^2 = \frac{e'e}{n-k}$$

IN THE 2-VARIABLE REGRESSION

$$(X'X)^{-1} = \frac{1}{n \sum x_i^2 - (\sum x_i)^2} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix}$$

THUS
$$s^{22} = \frac{1}{\sum x_i^2 - n\bar{x}^2} = \frac{1}{\sum (x_i - \bar{x})^2}$$

THUS
$$t = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{\hat{\sigma}^2 / \sum (x_i - \bar{x})^2}} \sim t_{n-2}$$