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STATISTICAL PROPERTIES OF LEAST SQUARES ESTIMATORS

CONSIDER THE MODEL: $y = X\beta + u$

HERE u IS A RANDOM VECTOR WITH

$E(u) = \underline{0}$; $V(u) = E(uu') = \sigma^2 I$

2-VARIABLE REGRESSION:

$$Y_i = \beta_1 + \beta_2 X_i + u_i \qquad (i = 1, 2, \ldots, n)$$

$$E(u_i) = 0 ; \quad \text{var}(u_i) = \sigma^2 ; \quad E(u_i u_j) = 0 \quad (i \neq j)$$

THEN $E(Y_i | X = X_i) = \beta_1 + \beta_2 X_i + E(u_i | X_i)$

$= \beta_1 + \beta_2 X_i$

THIS IS THE REGRESSION / CONDITIONAL EXPECTATION FUNCTION. HERE WE ASSUME THAT X_i & u_i ARE INDEPENDENT.

$$\hat{\beta}_2 = \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2}$$

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

$$\bar{Y} = \beta_1 + \beta_2 \bar{X} + \bar{u}$$

$$Y_i - \bar{Y} = \beta_2 (X_i - \bar{X}) + (u_i - \bar{u})$$

NOTE: $E(u_i) = 0$ DOES NOT MEAN THAT $\bar{u} = 0$.

$$\hat{\beta}_2 = \frac{\sum(X_i - \bar{X})[\beta_2 (X_i - \bar{X}) + (u_i - \bar{u})]}{\sum(X_i - \bar{X})^2}$$

$$= \frac{\beta_2 \sum(X_i - \bar{X})^2 + \sum(X_i - \bar{X})(u_i - \bar{u})}{\sum(X_i - \bar{X})^2}$$

$$= \beta_2 + \frac{\sum(X_i - \bar{X}) u_i}{\sum(X_i - \bar{X})^2}$$

$$= \beta_2 + \frac{(X_1 - \bar{X})u_1 + (X_2 - \bar{X})u_2 + \cdots + (X_n - \bar{X})u_n}{\sum(X_i - \bar{X})^2}$$

DEFINE $\frac{(x_i - \bar{x})}{\sum (x_i - \bar{x})^2} = \omega_i$

$$\beta_2 + (\omega_1 u_1 + \omega_2 u_2 + \cdots + \omega_n u_n)$$

$$E(\hat{\beta}_2) = \beta_2 + \omega_1 E(u_1) + \omega_2 E(u_2) + \cdots + \omega_n E(u_n)$$

$$= \beta_2 \quad (\text{BECAUSE } E(u_i) = 0 \text{ FOR ALL } i).$$

$$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X}$$

$$= \beta_1 + \beta_2 \bar{X} + \bar{u} - \hat{\beta}_2 \bar{X}$$

$$= \beta_1 + (\beta_2 - \hat{\beta}_2)\bar{X} + \bar{u}$$

$$E(\hat{\beta}_1) = \beta_1 - \bar{X} E(\hat{\beta}_2 - \beta_2) + E(\bar{u})$$

$$= \beta_1 . \quad (\text{B/c } E(\hat{\beta}_2) = \beta_2 \ \& \ E(\bar{u}) = 0).$$

THUS BOTH $\hat{\beta}_1$ & $\hat{\beta}_2$ ARE UNBIASED ESTIMATORS.

<u>VARIANCE OF</u> $\hat{\beta}_2$: $\mathrm{Var}(\hat{\beta}_2) = E(\hat{\beta}_2 - \beta_2)^2$

$$= E\left[(\omega_1 u_1 + \omega_2 u_2 + \cdots + \omega_n u_n)^2\right]$$

$$= E\left[\omega_1^2 u_1^2 + \omega_2^2 u_2^2 + \cdots + \omega_n^2 u_n^2 + 2\omega_1\omega_2 u_1 u_2 + \cdots + 2\omega_{n-1}\omega_n u_{n-1} u_n\right]$$

$$= \omega_1^2 E(u_1^2) + \omega_2^2 E(u_2^2) + \cdots + \omega_n^2 E(u_n^2)$$

[NOTE THE CROSS PRODUCTS DISAPPEAR B/c $E(u_i u_j) = 0$]

$$= \left[\omega_1^2 + \omega_2^2 + \cdots + \omega_n^2\right]\sigma^2 = \sigma^2 \sum \omega_i^2$$

BUT $\omega_i^2 = \frac{(x_i - \bar{x})^2}{\left[\sum (x_i - \bar{x})^2\right]^2} \Rightarrow \sum \omega_i^2 = \frac{\sum (x_i - \bar{x})^2}{\left[\sum (x_i - \bar{x})^2\right]^2}$

HENCE $\boxed{\mathrm{VAR}(\hat{\beta}_2) = \frac{\sigma^2}{\sum (x_i - \bar{x})^2}}$

$$\hat{\beta}_1 = \beta_1 - \bar{X}(\hat{\beta}_2 - \beta_2) + \bar{u}$$
$$= \beta_1 + \frac{1}{n}\sum u_i - \bar{X}\sum \omega_i u_i$$
$$= \beta_1 + \left[\sum\left(\frac{1}{n} - \omega_i \bar{X}\right)\right] u_i$$
$$\hat{\beta}_1 - \beta_1 = \sum\left(\frac{1}{n} - \omega_i \bar{X}\right) u_i$$
$$\text{VAR}(\hat{\beta}_1) = E\left[(\hat{\beta}_1 - \beta_1)^2\right]$$
$$= E\left\{\sum\left[\left(\frac{1}{n} - \omega_i \bar{X}\right) u_i\right]^2\right\}$$
$$\left(\frac{1}{n} - \omega_i\right)^2 = \left(\frac{1}{n}\right)^2 + \omega_i^2 \bar{X}^2 - 2\frac{\omega_i}{n}\bar{X}$$
$$\sum\left(\frac{1}{n} - \omega_i \bar{X}\right)^2 = \frac{1}{n} + \left(\sum \omega_i^2\right)\bar{X}^2$$
$$= \frac{1}{n} + \frac{\bar{X}^2}{\sum(X_i - \bar{X})^2}$$

THUS $\boxed{\text{VAR}(\hat{\beta}_1) = \sigma^2\left[\frac{1}{n} + \frac{\bar{X}^2}{\sum(X_i - \bar{X})^2}\right]}$

MULTIPLE REGRESSION

$$y = X\beta + u \quad ; \quad \hat{\beta} = (X'X)^{-1}X'y$$
$$\hat{\beta} = (X'X)^{-1}X'(X\beta + u) = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'u$$
$$\hat{\beta} = \beta + (X'X)^{-1}X'u$$
$$E(\hat{\beta}) = \beta + (X'X)^{-1}X'E(u) = \beta.$$
$$V(\hat{\beta}) = E\left[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'\right]$$
$$= E\left[(X'X)^{-1}X'uu'X(X'X)^{-1}\right] = (X'X)^{-1}X'E(uu')X(X'X)^{-1}$$
$$= (X'X)^{-1}X'\sigma^2 I X(X'X)^{-1} = \sigma^2(X'X)^{-1}X'X(X'X)^{-1} = \sigma^2(X'X)^{-1}.$$

IN THE 2-VARIABLE CASE,

$$X'X = \begin{bmatrix} n & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix} ; \quad |X'X| = n\Sigma x_i^2 - (\Sigma x_i)^2$$
$$= n\left[\Sigma x_i^2 - n\bar{x}^2\right]$$

$$(X'X)^{-1} = \frac{1}{n(\Sigma x_i^2 - n\bar{x}^2)} \begin{bmatrix} \Sigma x_i^2 & -\Sigma x_i \\ -\Sigma x_i & n \end{bmatrix}$$

$$V(\hat{\beta}) = \sigma^2 (X'X)^{-1}$$

$$\Rightarrow \quad Var(\hat{\beta}_2) = \frac{\sigma^2 n}{n\left[\Sigma x_i^2 - n\bar{x}^2\right]} = \frac{\sigma^2}{\Sigma x_i^2 - n\bar{x}^2} = \frac{\sigma^2}{\Sigma(x_i - \bar{x})^2}$$

$$Var(\hat{\beta}_1) = \frac{\sigma^2 \Sigma x_i^2}{n\left[\Sigma(x_i - \bar{x})^2\right]}$$

$$= \frac{\sigma^2}{n}\left[\frac{\Sigma x_i^2 - n\bar{x}^2 + n\bar{x}^2}{\Sigma(x_i - \bar{x})^2}\right]$$

$$= \frac{\sigma^2}{n}\left[1 + \frac{n\bar{x}^2}{\Sigma(x_i - \bar{x})^2}\right]$$

EXERCISE: FIND $Cov(\hat{\beta}_1, \hat{\beta}_2)$

ESTIMATOR OF σ^2:

$$e = y - \hat{y} = y - X\hat{\beta} = y - X(X'X)^{-1}X'y = My$$
$$= M(X\beta + u) = MX\beta + Mu = Mu$$

$$e'e = u'M'Mu = u'Mu$$

$$e'e = tr\left[u'Mu\right] = tr\left[Muu'\right]$$

$$E\left[e'e\right] = E\left[tr(Muu')\right] = tr\left[M\,E(uu')\right]$$
$$= \sigma^2 tr\left[MI\right] = \sigma^2 tr(M)$$

$$M = I - X(X'X)^{-1}X'$$

$$\mathrm{tr}(M) = \mathrm{tr}[I_n] - \mathrm{tr}[X(X'X)^{-1}X']$$

$$= n - \mathrm{tr}[(X'X)^{-1}X'X]$$

$$= n - \mathrm{tr}[I_k] = n-k.$$

THUS $E[e'e] = (n-k)\sigma^2$

$$E\left[\frac{e'e}{n-k}\right] = \sigma^2$$

DEFINE $\hat{\sigma}^2 = \frac{e'e}{n-k} = \frac{ESS}{n-k}$

THEN $E[\hat{\sigma}^2] = \frac{1}{n-k}E[e'e] = \frac{\sigma^2(n-k)}{n-k} = \sigma^2$

HENCE $\hat{\sigma}^2 = \frac{e'e}{n-k}$ IS AN UNBIASED ESTIMATOR OF σ^2.

IN ANY EMPIRICAL APPLICATION WE USE THE ERROR SUM OF SQUARES DIVIDED BY $(n-k)$ AS AN ESTIMATOR OF σ^2.

THUS $\widehat{\mathrm{Var}}(\hat{\beta}_2) = \frac{\hat{\sigma}^2}{\sum(X_i - \bar{X})^2} = \frac{ESS}{(n-2)\sum(X_i-\bar{X})^2}$.