

MAXIMUM LIKELIHOOD ESTIMATORS & HYPOTHESIS TESTING

PRELIMINARIES

CONSIDER A RANDOM VARIABLE y AND A SAMPLE OF n OBSERVATIONS:

$$Y = \{y_1, y_2, \dots, y_n\}$$

SUPPOSE THAT THE OBSERVATIONS ARE INDEPENDENTLY DRAWN FROM SOME PROBABILITY DISTRIBUTION WITH PARAMETER θ . FOR EXAMPLE THE DISTRIBUTION MAY BE THE NORMAL DISTRIBUTION WITH MEAN μ & VARIANCE σ^2 . THEN THE JOINT PDF OF THE SAMPLE IS

$$\begin{aligned} f(y_1, y_2, \dots, y_n; \mu, \sigma^2) &= \prod_{i=1}^n \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right) e^{-\frac{(y_i - \mu)^2}{2\sigma^2}} \\ &= (2\pi)^{-n/2} (\sigma^2)^{-n/2} e^{-\sum_{i=1}^n \frac{(y_i - \mu)^2}{2\sigma^2}} \end{aligned}$$

CLEARLY, THIS JOINT PDF IS A FUNCTION OF THE SAMPLE Y & VARIES FROM SAMPLE TO SAMPLE.

NOW CONSIDER THE FOLLOWING QUESTION. WE KNOW THAT THE SAMPLE OBSERVATIONS CAME FROM SOME NORMAL DISTRIBUTION. OUR OBJECTIVE IS TO DETERMINE WHICH PARTICULAR DISTRIBUTION (i.e. WHICH PAIR OF μ, σ^2) IS MOST LIKELY TO HAVE GENERATED THIS SAMPLE DATA SET.

FOR THIS WE WRITE THE SAME JOINT PDF AS A FUNCTION OF THE PARAMETERS μ & σ^2 :

$$L(\mu, \sigma^2; y_1, y_2, \dots, y_n)$$

THIS IS KNOWN AS THE LIKELIHOOD FUNCTION. THE MAXIMUM LIKELIHOOD PROCEDURE MAXIMIZES THE LIKELIHOOD FUNCTION L WITH RESPECT TO THE ~~LIKELI~~ PARAMETERS (μ, σ^2) .

THE 2-VARIABLE REGRESSION

$$y_i = \beta_1 + \beta_2 x_i + u_i$$

$$u_i \sim N(0, \sigma^2)$$

$$u_i = y_i - \beta_1 - \beta_2 x_i$$

RECALL $p(y_i) = p(u_i) \cdot \left| \frac{\partial u_i}{\partial y_i} \right|$

$$\frac{\partial u_i}{\partial y_i} = 1$$

$$p(y_i) = p(y_i - \beta_1 - \beta_2 x_i)$$

$$p(u_1, u_2, \dots, u_n) = \left(\frac{1}{\sqrt{2\pi}\sigma^2} \right)^n e^{-\frac{\sum u_i^2}{2\sigma^2}}$$

$$p(y_1, y_2, \dots, y_n) = (2\pi)^{-n/2} (\sigma^2)^{-n/2} e^{-\frac{\sum (y_i - \beta_1 - \beta_2 x_i)^2}{2\sigma^2}}$$

$$= L(\beta_1, \beta_2, \sigma^2; y_1, y_2, \dots, y_n)$$

NEXT CONSIDER THE LOG LIKELIHOOD FUNCTION

$$\ln L = -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln \sigma^2 - \frac{1}{2\sigma^2} \sum (y_i - \beta_1 - \beta_2 x_i)^2$$

MAXIMIZING $\ln L$ IS EQUIVALENT TO MAXIMIZING L .

FOCs :

$$\frac{\partial \ln L}{\partial \beta_1} = \frac{1}{\sigma^2} \sum (y_i - \beta_1 - \beta_2 x_i) = 0$$

$$\Rightarrow \boxed{\frac{\sum (y_i - \beta_1 - \beta_2 x_i)}{\sigma^2} = 0} \quad \text{(A)}$$

$$\frac{\partial \ln L}{\partial \beta_2} = \frac{\sum x_i (y_i - \beta_1 - \beta_2 x_i)}{\sigma^2} = 0$$

$$\Rightarrow \boxed{\frac{\sum x_i (y_i - \beta_1 - \beta_2 x_i)}{\sigma^2} = 0} \quad \text{(B)}$$

$$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum (y_i - \beta_1 - \beta_2 x_i)^2 = 0$$

$$\Rightarrow \boxed{\frac{\sum (y_i - \beta_1 - \beta_2 x_i)^2}{\sigma^2} = n} \quad \text{(C)}$$

NOTE

$$\text{(A)} \Rightarrow \sum (y_i - \beta_1 - \beta_2 x_i) = 0$$

$$\text{(B)} \Rightarrow \sum x_i (y_i - \beta_1 - \beta_2 x_i) = 0$$

THESE ARE EXACTLY THE SAME AS THE FOCs FOR OLS. THUS

SOLUTION OF (A) + (B) YIELDS

$$\beta_2^* = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} \quad \&$$

$$\beta_1^* = \bar{y} - \beta_2^* \bar{x}$$

THAT IS β_1^* & β_2^* ARE THE SAME AS $(\hat{\beta}_1, \hat{\beta}_2)$. HENCE

$$y_i - \beta_1^* - \beta_2^* x_i = y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i = e_i$$

(THE OLS RESIDUAL)

HENCE, BY VARTUE OF (A) & (B),

$$\textcircled{c} \Rightarrow \frac{\sum e_i^2}{\sigma_v^2} = n \quad \& \quad \sigma_v^2 = \frac{\sum e_i^2}{n}$$

NOTE THAT THE MLE OF σ^2 IS DIFFERENT FROM THE OLS BASED ESTIMATOR $\frac{1}{n} \sum e_i^2$.

$$\text{THUS, } \ln L^x = -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln\left(\frac{SSE}{n}\right) - \frac{1}{2} \frac{(SSE)}{\left(\frac{SSE}{n}\right)}$$

$$= -\frac{n}{2} \ln 2\pi - \frac{n}{2} - \frac{n}{2} \ln\left(\frac{SSE}{n}\right) = -\frac{n}{2} \ln 2\pi - \frac{n}{2} + \frac{n}{2} \ln n - \frac{n}{2} \ln S$$

DEFINE $\frac{SSE}{Q} = Q$ & $-\frac{n}{2} \ln 2\pi - \frac{n}{2} + \frac{n}{2} \ln n = c$

$$\text{THEN } \ln L^x = c - \frac{n}{2} \ln \frac{SSE}{n} = \boxed{c - \frac{n}{2} \ln Q}$$

THE LIKELIHOOD RATIO TEST:

LET $ESS_R = Q_R$ BE THE ERROR SUM OF SQUARES FROM THE RESTRICTED MODEL

& $ESS_U = Q_U$ BE THE ERROR SUM OF SQUARES FROM THE UNRESTRICTED MODEL

THEN

$$\ln L^*_R = c - \frac{n}{2} \ln Q_R$$

$$\ln L^*_U = c - \frac{n}{2} \ln Q_U$$

$$\begin{aligned} \ln L^*_R - \ln L^*_U &= -\frac{n}{2} [\ln Q_R - \ln Q_U] \\ &= -\frac{n}{2} \ln \left(\frac{Q_R}{Q_U} \right) \end{aligned}$$

THUS THE LIKELIHOOD RATIO IS

$$\frac{L^*_R}{L^*_U} = \left(\frac{Q_R}{Q_U} \right)^{-n/2} = \lambda$$

$-2 \ln \lambda = n (\ln Q_R - \ln Q_U) \sim \chi^2_p$
 WHERE $p = \#$ OF RESTRICTIONS.

CONSIDER THE RESTRICTION $\beta_2 = 0$
IN THE MODEL

$$y_i = \beta_1 + \beta_2 x_i + u_i$$

UNDER THE RESTRICTION,

$$y_i = \beta_1 + u_i$$

THEN $\hat{\beta}_1 = \bar{y}$ &

$$Q_R = (y_i - \bar{y})^2 \equiv S_{yy}$$

ALSO IN THE UNRESTRICTED MODEL

$$1 - R^2 = \frac{SSE}{SST} = \frac{Q_U}{Q_R}$$

THUS $\frac{Q_R}{Q_U} \equiv \frac{1}{1 - R^2}$

$$\& -2 \ln \lambda = -n \ln(1 - R^2) = n \ln \left(\frac{1}{1 - R^2} \right) \sim \chi^2_1$$

THE WALD TEST

EXCEPT IN SIMPLE CASE LIKE THE ONE CONSIDERED ABOVE THE LR TEST REQUIRES ONE TO ESTIMATE THE RESTRICTED & THE UNRESTRICTED MODELS TO GET $\hat{\beta}_R$ & $\hat{\beta}_U$ FOR COMPUTING THE TEST STATISTIC. THE WALD TEST IS BASED ON THE PROPERTY THAT WHEN $u \sim N(0, \Sigma)$ UNDER THE RESTRICTION $R\beta = q$,

$$(R\hat{\beta} - q)' V (R\hat{\beta} - q) \sim \chi^2_p \quad \text{WHERE}$$

$p = \#$ OF RESTRICTIONS.

IN THIS SIMPLE CASE, UNDER $H_0: \beta_2 = 0$

$$W = \frac{\hat{\beta}_2^2}{\text{Var } \hat{\beta}_2} \sim \chi_1^2$$

THUS

$$W = \frac{\hat{\beta}_2^2}{\frac{\sigma^2}{\sum (x_i - \bar{x})^2}} = \frac{\hat{\beta}_2^2 \sum (x_i - \bar{x})^2}{\sigma^2} \sim \chi_1^2$$

IF WE USE $\frac{\sum e_i^2}{n} = \frac{\sum e_i^2}{n}$ [THE MLE OF σ^2]

$$W = \left[\frac{\hat{\beta}_2^2 \sum (x_i - \bar{x})^2}{\sum e_i^2} \right] n$$

$$= \left[\frac{SSR}{SSE} \right] n = \left[\frac{SSR/SST}{SSE/SST} \right] n$$

$$W = \left[\frac{R^2}{1-R^2} \right] n \sim \chi_1^2$$

THE LAGRANGE MULTIPLIER TEST

IF WE USE THE MLE OF σ^2
FROM THE RESTRICTED MODEL,

$$LM = \frac{\hat{\beta}_2^2 \sum (x_i - \bar{x})^2}{\sigma_{KR}^2} = \frac{\hat{\beta}_2^2 \sum (x_i - \bar{x})^2}{\sum (y_i - \bar{y})^2 / n}$$

$$= \frac{SSR}{SST} \cdot n = \boxed{n R^2 \sim \chi_1^2}$$

NOTE THAT

$$LR = n \ln \left(\frac{1}{1-R^2} \right)$$

$$W = nR^2 / (1-R^2)$$

$$LM = nR^2$$

CLEARLY, BECAUSE $0 \leq R^2 \leq 1$,

$$0 \leq 1-R^2 \leq 1 \Rightarrow \frac{1}{1-R^2} \geq 1$$

THUS, $W \geq LM$

FURTHER, GIVE THAT

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \text{FOR } x \geq 0$$

$$e^x \geq 1+x \Rightarrow x \geq \ln(1+x)$$

$$\frac{LR}{n} = \ln \left(\frac{1}{1-R^2} \right)$$

$$\frac{W}{n} = \frac{R^2}{1-R^2}$$

$$\frac{W}{n} + 1 = \frac{1}{1-R^2}$$

$$\ln \left(\frac{W}{n} + 1 \right) = \ln \frac{1}{1-R^2}$$

THUS $\frac{W}{n} \geq \ln \left(\frac{W}{n} + 1 \right)$

$$\Rightarrow \frac{W}{n} \geq \frac{LR}{n} \Rightarrow$$

$W \geq LR$

FINALLY

$$\ln(1+x) \geq \frac{x}{1+x}$$

DEFINE

$$\frac{LM}{n} = \frac{W/n}{1+W/n}$$

$$\frac{LR}{n} = \ln\left(1 + \frac{W}{n}\right)$$

HENCE

$$\frac{LR}{n} \geq \frac{LM}{n}$$

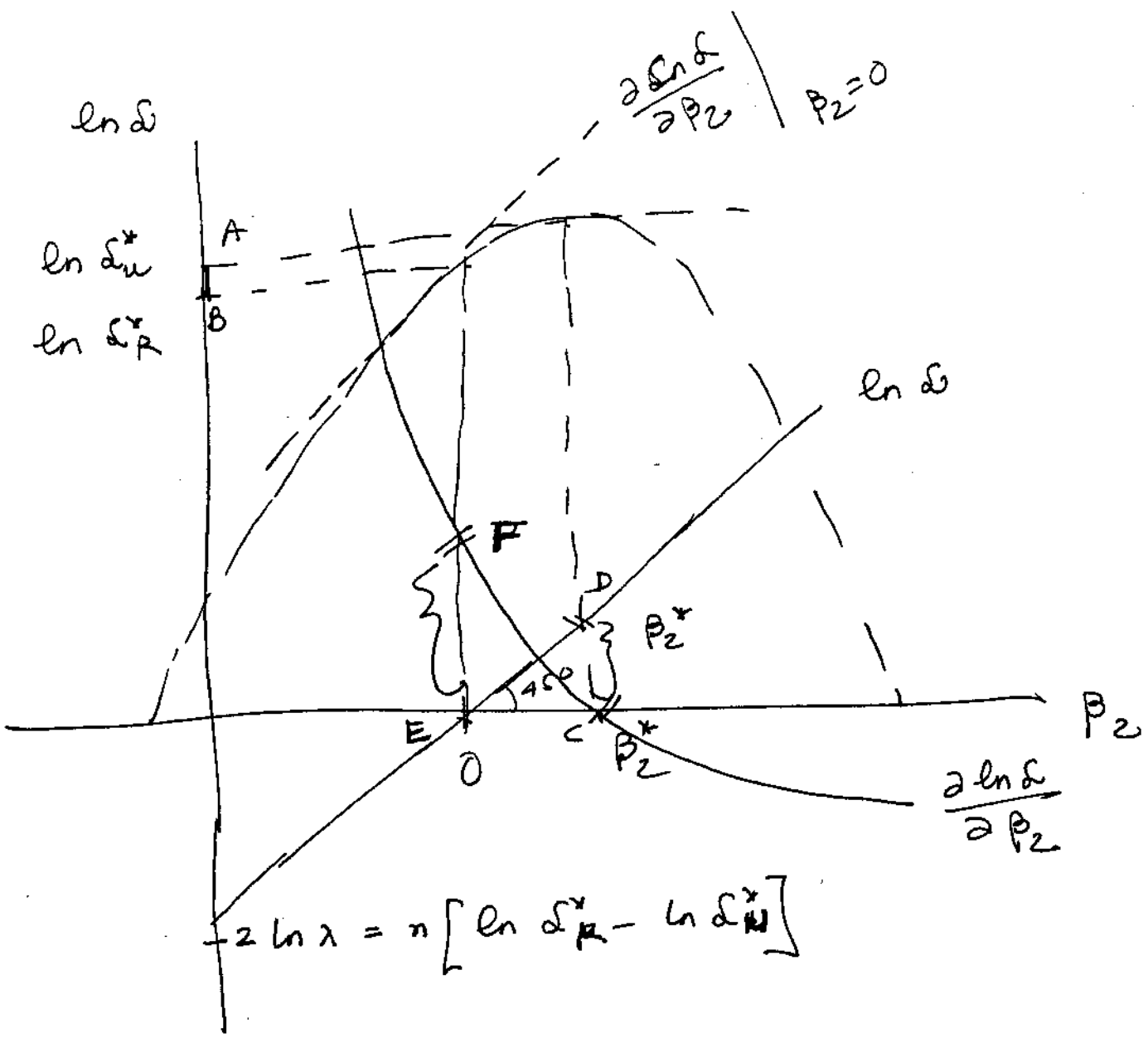
THUS

$$\frac{W}{n} \geq \frac{LR}{n} \geq \frac{LM}{n}$$

 \Rightarrow

$$W \geq LR \geq LM$$

BECAUSE EACH OF THE TEST STATISTICS IS χ^2_{1}
 IF $LM > \chi^2_{1\alpha}$, BOTH LR & W
 ARE ALSO SIGNIFICANT.



$$-2 \ln \lambda = n [\ln S^*_{R} - \ln S^*_{W}]$$

LR : IS AB SIGNIFICANTLY DIFFERENT FROM 0 ?

W : IS CD SIGNIFICANTLY DIFFERENT FROM 0 ?

LM : IS EF SIGNIFICANTLY DIFFERENT FROM 0 ?