

BIVARIATE RANDOM VECTOR

2 DECKS OF CARDS (4 CARDS IN EACH)
 NUMBERED 1, 2, 3, 4

D_1 : NUMBER DRAWN FROM DECK 1

D_2 : NUMBER DRAWN FROM DECK 2

2 CARDS ARE DRAWN - 1 FROM EACH GROUP.

$$(x = D_1; y = |D_1 - D_2|).$$

SAMPLE SPACE

	$D_2 = 1$	2	3	4
$D_1 = 1$	1,1	1,2	1,3	1,4
2	2,1	2,2	2,3	2,4
3	3,1	3,2	3,3	3,4
4	4,1	4,2	4,3	4,4

MAPPING FROM SAMPLE SPACE ON TO R^2

$$\begin{array}{cccc}
 (x=1, y=0) & (x=1, y=1) & (x=1, y=2) & (x=1, y=3) \\
 (x=2, y=1) & (x=2, y=0) & (x=2, y=1) & (x=2, y=2) \\
 (x=3, y=2) & (x=3, y=1) & (x=3, y=0) & (x=3, y=1) \\
 (x=4, y=3) & (x=4, y=2) & (x=4, y=1) & (x=4, y=0)
 \end{array}$$

BIVARIATE PROBABILITY DISTRIBUTION

	y=0	1	2	3	P(x)
x=1	1/16	1/16	1/16	1/16	1/4
=2	1/16	2/16	1/16	0	1/4
=3	1/16	2/16	1/16	0	1/4
=4	1/16	1/16	1/16	1/16	1/4
P(y)	1/4	6/16	1/4	1/8	

MARGINAL PROBABILITY DISTRIBUTION

x =	1	2	3	4
P(x)	1/4	1/4	1/4	1/4

$E(x) = 2\frac{1}{2}$

y =	0	1	2	3
P(y)	1/4	3/8	1/4	1/8

$E(y) = 1\frac{1}{4}$

CONDITIONAL PROBABILITY

411
1-3

DISTRIBUTION

$$P(y=b|x=a) = \frac{P(y=b, x=a)}{P(x=a)}$$

$y =$	0	1	2	3
$P(y x=1)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

$$E(y|x=1) = 1\frac{1}{2}$$

$y =$	0	1	2	3
$P(y x=2)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0

$$E(y|x=2) = 1$$

$y =$	0	1	2	3
$P(y x=3)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0

$$E(y|x=3) = 1$$

$y =$	0	1	2	3
$P(y x=4)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

$$E(y|x=4) = 1\frac{1}{2}$$

NOTE THAT THE CONDITIONAL EXPECTATION
 $E(y|x)$ IS A FUNCTION OF x .

THIS IS THE REGRESSION FUNCTION
 OF y ON x .

BIVARIATE NORMAL DISTRIBUTION

2 INDEPENDENT RANDOM VARIABLES u_1 & u_2

$$f(u_1) = \frac{1}{\sqrt{2\pi}} e^{-u_1^2/2}$$

$$f(u_2) = \frac{1}{\sqrt{2\pi}} e^{-u_2^2/2}$$

THE JOINT PDF

$$R(u_1, u_2) = f(u_1) \cdot f(u_2)$$

$$= \frac{1}{2\pi} e^{-\frac{(u_1^2 + u_2^2)}{2}}$$

DEFINE $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

THEN $R(u) = \frac{1}{2\pi} e^{-\frac{u'u}{2}}$

NOW DEFINE

$$x_1 = a_{11}u_1 + a_{12}u_2 + b_1$$

$$x_2 = a_{21}u_1 + a_{22}u_2 + b_2$$

$$x = Au + b$$

$$E(x) = AE(u) + b = b$$

$$\Sigma = E[(x-b)(x-b)']$$

$$x-b = Au$$

$$\Sigma = E[Auu'A'] = AE(uu')A'$$

$$E(uu') = I$$

$$\Rightarrow \Sigma = AA'$$

$$u = \bar{A}'(x-b)$$

$$u'u = (x-b)'\bar{A}'\bar{A}'^{-1}(x-b)$$

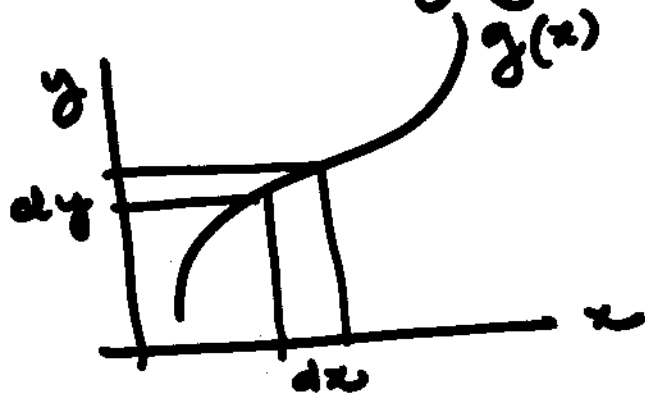
$$= (x-b)'(AA')^{-1}(x-b)$$

$$= (x-b)'\bar{\Sigma}^{-1}(x-b)$$

DIGRESSION

SUPPOSE x HAS THE PDF $f(x)$
 & $y = g(x)$ WHERE $g(\cdot)$ IS A
 MONOTONIC FUNCTION.

$$x = \bar{g}^{-1}(y)$$



$$Pr(x \in dx) = Pr(y \in dy)$$

$$Pr(x \in dx) = f(x) dx$$

SUPPOSE $h(y)$ IS THE pdf OF y . ⁴¹¹ (1-6)

THEN $\Pr(y \in dy) = h(y) dy$
 $= f(x) dx = \Pr(x \in dx)$

$$\Rightarrow h(y) = f(x) \cdot \frac{dx}{dy}$$
$$= f(x) \frac{1}{dy/dx}$$

BUT dy/dx MAY BE NEGATIVE.

HENCE TO ENSURE THAT THE pdf $h(y)$ IS NON-NEGATIVE, WE TAKE $|dy/dx|$.

THUS $h(y) = f(x) \cdot \left| \left(\frac{dy}{dx} \right)^{-1} \right|$.

EXAMPLE

$x \sim N(0,1)$. $y = \sqrt{x} \Rightarrow x = y^2$
 $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$$h(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^4}{2}} |2y|$$

$$h(y) = \left(\sqrt{\frac{2}{\pi}} \right) e^{-\frac{y^4}{2}} |y|$$

FOR THE BIVARIATE DISTRIBUTION

$$x = Au + b$$

THE JACOBIAN MATRIX IS

$$J = \left[\frac{\partial x}{\partial u} \right] = \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} \end{bmatrix}$$

$$h(x_1, x_2) = f(u_1, u_2) | | J | |^{-1}$$

IN OUR CASE $|J| = |A|$.

RECALL THAT $\Sigma = AA'$

$$|\Sigma| = |AA'| = (|A|)^2$$

$$|A| = |\Sigma|^{1/2}$$

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} \quad |\Sigma| = \sigma_1^2 \sigma_2^2 - \sigma_{12}^2 = \sigma_1^2 \sigma_2^2 (1 - \rho^2)$$

$$\rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$$

$$= \frac{1}{2} (x-b)' \Sigma^{-1} (x-b)$$

$$h(x_1, x_2) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1 - \rho^2}}$$

DEFINE

$$Q \equiv \frac{1}{2} (x-b)' \Sigma^{-1} (x-b).$$

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}$$

$$\Sigma^{-1} = \begin{pmatrix} \sigma_2^2 - \sigma_{12} & \\ -\sigma_{21} & \sigma_1^2 \end{pmatrix} / |\Sigma|$$

$$= \frac{1}{\sigma_1^2 \sigma_2^2 (1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{21} & \sigma_1^2 \end{bmatrix}$$

$$Q = \frac{1}{2 \sigma_1^2 \sigma_2^2 (1-\rho^2)} \begin{bmatrix} (x_1 - b_1) & (x_2 - b_2) \end{bmatrix} \begin{bmatrix} \sigma_2^2 - \sigma_{12} & \\ -\sigma_{21} & \sigma_1^2 \end{bmatrix} \begin{bmatrix} x_1 - b_1 \\ x_2 - b_2 \end{bmatrix}$$

$$= \frac{1}{2 \sigma_2^2 (1-\rho^2)} \left[(x_2 - b_2)^2 - 2 \frac{\sigma_{12}}{\sigma_1^2} (x_1 - b_1) (x_2 - b_2) + (x_1 - b_1)^2 \frac{\sigma_2^2}{\sigma_1^2} \right]$$

$$Q = \frac{1}{2 \sigma_2^2 (1-\rho^2)} \left[x_2 - \left(b_2 - \frac{\sigma_{12}}{\sigma_1^2} b_1 \right) - \frac{\sigma_{12}}{\sigma_1^2} x_1 \right]^2 + \frac{1}{2 \sigma_1^2} (x_1 - b_1)^2.$$

HENCE

$$f(x_1, x_2) = \left[\frac{1}{\sqrt{2\pi} \sigma_1} e^{-\frac{(x_1 - b_1)^2}{2\sigma_1^2}} \right] \cdot \left[\frac{1}{\sqrt{2\pi(1-\rho^2)} \cdot \sigma_2} e^{-\frac{1}{2\sigma_2^2(1-\rho^2)} \left[x_2 - \left(b_2 - \frac{\sigma_{12}}{\sigma_1^2} b_1 \right) - \frac{\sigma_{12}}{\sigma_1^2} x_1 \right]^2} \right]$$

THEN

$$f(x_2|x_1) = \frac{f(x_1, x_2)}{f(x_1)}$$

$$= \frac{1}{\sqrt{2\pi(1-\rho^2)} \sigma_2} e^{-\frac{1}{2\sigma_2^2(1-\rho^2)} \left[x_2 - \left(b_2 - \frac{\sigma_{12}}{\sigma_1^2} b_1 \right) - \frac{\sigma_{12}}{\sigma_1^2} x_1 \right]^2}$$

DEFINE $\sigma_2^2(1-\rho^2) = \hat{\sigma}_2^2$

& $b_2 - \frac{\sigma_{12}}{\sigma_1^2} b_1 + \frac{\sigma_{12}}{\sigma_1^2} x_1 = \hat{b}_2(x_1)$.

THEN $f(x_2|x_1) \sim N(\hat{b}_2(x_1), \hat{\sigma}_2^2)$.

THUS THE CONDITIONAL DISTRIBUTION OF x_2 GIVEN x_1 IS NORMAL

WITH

$$E(x_2 | x_1) = \alpha + \beta x_1$$

$$\alpha = b_2 - \frac{\sigma_{12}}{\sigma_1^2} b_1$$

$$\beta = \frac{\sigma_{12}}{\sigma_1^2}$$

NOTE: (1) THE CONDITIONAL MEAN OF x_2 GIVEN x_1 IS A FUNCTION OF x_1 .

(2) THE CONDITIONAL VARIANCE OF x_2 GIVEN x_1 IS NOT A FUNCTION OF x_2 . THE VARIANCE IS A CONSTANT. THIS

PROPERTY IS KNOWN AS HOMOSCEDASTICITY.